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FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2024

(CBCSS)

Mathematics

MTH4E09—DIFFERENTIAL GEOMETRY

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Sketch the level sets $f^{-1}(-1), f^{-1}(0)$ for $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots + x_n^2 - x_{n+1}^2, n = 1$.

2. Use Lagrange's multiplier method to find three positive numbers whose sum is 36 and their product is as large as possible.

3. Let S be an n-plane $a_1 x_1 + \ldots + a_{n+1}, x_{n+1} = b$, let $p, q \in S$ and $\mathbf{v} = (p, v) \in S_p$. If α is any parametrized curve in S from p to q, find the parallel transport of $\mathbf{v}$ along α to q.

4. Show that $\alpha(t) = (\cos at) e_1 + (\sin at) e_2$ is a geodesic in $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$ for some $a \in \mathbb{R}$ and for some orthogonal pair of unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$.

5. Compute $\nabla_{\mathbf{v}} f$, where $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}_p^3, p \in \mathbb{R}^3$ is $f(x_1, x_2, x_3) = x_1 x_2 x_3^2$, $\mathbf{v} = (1, 1, 1, a, b, c)$.

6. Compute the line integral $\int_C (-x_2 dx_1 + x_1 dx_2)$ where C is the ellipse $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, oriented by its inward normal.

7. Show that the integral of an exact 1-form over a closed curve is zero.

Turn over

8. Evaluate the normal curvature of an n-sphere S of radius r at $p \in S$ in the direct

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Part B

Answer **two** *questions from each unit.*

Each question carries a weightage of 2.

UNIT I

9. Determine completeness of the vector field $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$ on $U = \mathbb{R}^2 - \{($

10. Show that if S is a connected n-surface in $\mathbb{R}^{n+1}$ and $g : S \to \mathbb{R}$ is smooth and takes values 1 and -1, then g is constant.

11. Show that the set of all unit vectors at all points of $\mathbb{R}^2$ is a 3-surface in $\mathbb{R}^2$.

UNIT II

12. Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $\beta : I \to S$ be a geodesic in S with $\beta(t_0) =$ $\dot{\beta}(t_0) = \dot{\beta}(0)$ for some $t_0 \in I$, $t_0 \neq 0$. Show that β is periodic by showing that $\beta(t + t_0) =$ all t such that both t and $t + t_0 \in I$.

13. If X and Y are smooth vector fields on an n-surface S with $\mathbf{v} \in S_p$, $p \in S$, then sh $\nabla_{\mathbf{v}}(\mathbf{X} \cdot \mathbf{Y}) = (\nabla_{\mathbf{v}}\mathbf{X}) \cdot \mathbf{Y}(p) + \mathbf{X}(p) \cdot (\nabla_{\mathbf{v}}\mathbf{Y})$.

14. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by $\eta = \dfrac{-x_2}{x_1^2 + x_2^2}dx_1 + \dfrac{x_1}{x_1^2 + x_2^2}dx_2$ and let C be the $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1$. Show that η over C is not exact.

15. Let S be the hyperboloid in $\mathbb{R}^3$ given by $-x_1^2 + x_2^2 + x_3^2 = 1$. Find the normal curvature of S at $(0, 0, 1)$ in the direction of a unit vector $\mathbf{v} \in S_p$, oriented by the normal $N(p) = \left(p, \frac{-x_1}{\|p\|}, \frac{x_2}{\|p\|}, \frac{x_3}{\|p\|}\right)$ for $p = (x_1, x_2, x_3) \in S$.

16. Find the Guassian curvature of a cylinder over a plane curve.

17. Describe the parametrized n-plane and verify that it is regular.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) State and prove existence and uniqueness of maximal integral curve of a smooth tangent vector field X on an n-surface $S \subseteq \mathbb{R}^{n+1}$ and through $p \in S$.

(b) State and prove Lagrange's Multiplier theorem.

19. (a) Show that the Weingarten map of an n-surface S at $p \in S$ is self adjoint.

(b) Find global parametrization and curvature of $ax_1 + bx_2 = c$, $(a, b) \neq (0, 0)$, oriented by the outward normal.

20. (a) For each 1-form w on U (U open in $\mathbb{R}^{n+1}$), show that there exist unique functions $f_i : U \to \mathbb{R}$ $(i \in \{1, 2, ..., n+1\})$ such that $w = \sum_{i=1}^{n+1} f_i \, dx_i$. Also show that w is smooth if and only if each f_i is smooth.

Turn over

(b) Find the Guassian curvature of the parametrized torus in $\mathbb{R}^3$ given by :

$$\psi(\theta, \phi) = \left(\left(a + b\cos\phi\right)\cos\theta, \left(a + b\cos\phi\right)\sin\theta, b\sin\phi\right) \text{ where, } a > b > 0.$$

21. Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $p \in S$. Show that there exists an open set V in $\mathbb{R}^{n+1}$ and a parametrized n-surface $\phi : U \to \mathbb{R}^{n+1}$ such that ϕ is a one to one map onto $V \cap S$.

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