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FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2024

(CBCSS)

Mathematics

MTH4E08—COMMUTATIVE ALGEBRA

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*

Each question carries weightage 1.

1. Find a non-zero nilpotent element in the ring $\mathbb{Z}_8$ of integers mod 8.

2. Let $\mathbb{Z}$ be the ring of integers. Let $a = \langle 4 \rangle$ and $b = \langle 6 \rangle$ be ideals of $\mathbb{Z}$. Find a generator of the ideal $a + b$.

3. Let $\mathbb{Z}_2$ be the ring of integers mod 2 and $\mathbb{Z}_4$ be the cyclic group of order 4. Describe an action of $\mathbb{Z}_2$ on $\mathbb{Z}_4$ making it an $\mathbb{Z}_2$-module.

4. Show that in the ring of fractions $S^{-1}A$, for any $s \in S$, s/s is the identity element.

5. Let $\mathbb{Z}$ be the ring of integers and I be the ideal generated by 3 ans $S = \mathbb{Z} - I$. Verify whether $1/6 \in S^{-1}\mathbb{Z}$.

6. Let $A = k[x, y]$ be the polynomial ring over a field k and let $q = (x, y^2)$ be the ideal generated by x and y^2. Show that q is a primary ideal.

7. Verify whether $\frac{2}{3}$ is integral over $\mathbb{Z}$.

8. Show that the polynomial ring $k[x]$ over a field k is integrally closed.

(8 × 1 = 8 weightage)

Turn over

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Part B

*Answer any **two** questions from each unit.*
Each question carries weightage 2.

Unit I

9. Let x be a nilpotent element in a commutative ring A. Show that $1-x$ is a unit in A.

10. Let R be the nilradical of a ring A. Show that A/R has no nonzero nilpotent elements.

11. Let L be an A-module and M, N be submodules of L such that $N \subseteq M \subseteq L$. Show that (L/N)
is isomorphic to L/M.

Unit II

12. Let S be a multiplicatively closed subset of a commutative ring A. Consider ~ on A × S defi

$$(a, s) \sim (b, t) \text{ if } (at - bs)u = 0 \text{ for some } u \in S.$$

Show that ~ is an equivalence relation on A × S.

13. Let N, P be submodules of an A-module M and S be a multiplicatively closed sub-set of A
that $S^{-1}(N + P) = S^{-1}N + S^{-1}P$.

14. Let q be a primary ideal in a ring A. Show that the radical $r(q)$ of q is a prime ideal.

Unit III

15. Let A be a subring of a ring B and C be the set of all elements of B which are integral over A
that C is a subring of B.

16. Let A be an integral domain. Show that if A is integrally closed then A_p is integrally clo
every prime ideal p.

17. Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be an exact sequence of A-modules. Show that
Noetherian then M' and M'' are Noetherian.

(6 × 2 = 12 weig

Part C

Answer any **two** *questions.*
Each question carries weightage 5.

18. Let A be a non-zero commutative ring. Show that the following are equivalent.

    (a) A is a field.

    (b) The only ideals of A are (0) and (1).

    (c) Every homomorphism ϕ from A to a non-zero ring B is injective.

19. (a) Let M be an A-module. Show that M is finitely generated if and only if M is isomorphic to a quotient of A^n for some positive integer n.

    (b) Let M be a finitely generated A-module and R be the Jacobson radical of A. Show that $RM \neq M$.

20. (a) Let S be a multiplicatively closed subset of a ring A. Describe the elements and the ring structure of $S^{-1}A$.

    (b) Let A, B be rings and S be a multiplicative suset of A. Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for every $s \in S$. Show that there exists a unique homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$ where $f : A \to S^{-1}A$ is the natural inclusion.

21. (a) Define valuation ring.

    (b) Let B be an integral domain and K be its field of fractions. Prove that if B is a valuation ring over K then :

    (i) B is a local ring.

    (ii) If B' is a ring such that $B \subseteq B' \subseteq K$ then B' is a valuation ring of K.

    (iii) B is integrally closed in K.

(2 × 5 = 10 weightage)