D 101266 (Pages : 3) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2024

(CBCSS)

Mathematics

MTH4C15—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Define spectrum of a bounded operator.
2. Prove that for every compact operator $T, 0 \in \sigma(T)$.
3. If A and B are symmetric operators and AB = BA, then prove that AB is also symmetric.
4. If A is symmetric, then prove that $A^{2n} \geq 0$.
5. Prove that the Gelfand transform has the following property :

   $\left(\widehat{x_1 + x_2}\right) = \hat{x}_1(M) + \hat{x}_2(M)$.
6. State Gelfand-Mazur theorem.
7. Define regular point of an element in an algebra.
8. Define Wiener algebra.

(8 × 1 = 8 weightage)

Part B

Answer **six** *questions choosing two from each module.*
Each question has weightage 2.

Module I

9. Prove that every complex number λ with $|\lambda| > \|A\|$ is a regular point of the operator A.
10. Let E_1 be a closed subspace of E such that $E \neq E_1 \mapsto E \mapsto X$. Then prove that there exists $y_0 \in E$ with $\|y_0\| = 1$ and such that the distance of y_0 to E_1 satisfies $\text{dist}(y_0, E_1) \geq 1/2$.

Turn over

11. Find the spectrum of the operator D_w in l_2 which is defined by $D_w x = (w_1 x_1, w_2$ ...

$x = (x_1, x_2, \ldots)$.

Module II

12. If $A \geq 0$ and $\langle Ax, x \rangle = 0$ then prove that $Ax = 0$.
13. If P is an orthoprojection, then prove that $\operatorname{Im} P \perp \ker P$.
14. Let $\varphi(t) \in K[a,b]$. Then prove that there exists a sequence of polynomials $P_n(t) \searrow \varphi(t)$ for all $t \in [a,b]$.

Module III

15. Prove that every complete metric space M is a set of second category
16. If X^* is a separable space, then prove that X is also separable.
17. Prove that for every proper ideal $I \subseteq A$ there exists a maximal ideal M such that $I \subseteq M$

(6 × 2 = 12 ...

Part C

Answer **two** *questions.*
Each question has weightage 5.

18. State and prove the first Hilbert-Schmidt theorem.
19. (a) Let $A_0 \leq A_1 \leq \ldots \leq A_n \leq \ldots \leq A$. Then prove that there exists a strong limit of (A_n) ...
    (b) If $P_1 P_2 = P_1$, then prove that $E_1 \mapsto E_2$ and $P_1 \leq P_2$. Also, prove that $P_1 \leq P_2$ implies ... and $E_1 \mapsto E_2$.
20. If K is perfectly convex in a Bananch space X, then prove that :

$$\overset{o}{K} = \overset{c}{K} = \overset{\underline{c}}{K} = \left(\overset{\underline{o}}{K} \right).$$

21. Let $e = \{e_k\}_{k=1}^{\infty}$ a complete linearly independent system in a Banach space X. Then, prove that e is a basis of X if and only if the projections U_n defined by :

$$u_n\left(\sum_{k=1}^{m} a_k e_k\right) = \sum_{k=1}^{n} a_k e_k$$

for all $a_k \in F$ (if $m < n$ the values a_k for $k > m$ are assumed to be zero) on the dense subset $M = \text{span}\,\{e\}$ are uniformly bounded.

(2 × 5 = 10 weightage)