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FOURTH SEMESTER M.Sc. REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, APRIL 2022

April 2021 Session for SDE/Private Students

(CBCSS)

Mathematics

MTH 4E 08—COMMUTATIVE ALGEBRA

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

Covid Instructions are not applicable for Pvt/SDE students (April 2021 session)

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section/Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section/sub part/sub division may be ignored.*
4. *There will be an overall ceiling for each Section/Part that is equivalent to the maximum weightage of the Section/Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Let A be a ring $\neq 0$ such that the only ideals in A are 0 and (1). Prove that every homomorphism of A into a non-zero ring B is injective.
2. If M_1 and M_2 are submodules of an A-module M. Prove that $(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2)$.
3. Define an exact sequence of A-modules and A-homomorphisms.
4. If P is a prime ideal of a ring A, prove that A-P is a multiplicatively closed subset of A.
5. Let I be an ideal of a ring A such that $r(I)$ is maximal. Prove that I is a primary ideal.
6. Let B be a ring and A is a subring of B. Prove that the set of elements of B which are integral over A is a subring of B containing A.

Turn over

7. Define an integrally closed integral domain. Give an example.

8. Prove that in a Noetherian ring the nilradical is nilpotent.

(8 × 1 = 8 weight

Part B

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Let R denotes the Jacobson radical of a ring A. Prove that $x \in R \Leftrightarrow 1 - xy$ is a unit in all $y \in A$.

10. If N and P are submodules of an M-Module M. Prove that $(N : P) = \text{Ann}((N + P)/N)$.

11. Let M, N be A-modules. Prove that there is a unique isomorphism from $M \otimes N \rightarrow N$

Module II

12. Let M be an A-module and S a multiplicatively closed subset of A. Prove that the modules $S^{-1} M$ and $S^{-1} A \otimes_A M$ are isomorphic.

13. Let M be an A-module. Prove that M = 0 if and only if $M_n = 0$ for all maximal ideals m

14. Prove that the isolated primary components of a decomposable ideal I are uni determined by I.

Module III

15. State and prove the going up theorem.

16. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A-modules. Prove that M is Noet if and only if M′ and M″ Noetherian.

17. In an Artin ring prove that the nilradical is equal to the Jacobson radical.

(6 × 2 = 12 weig

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) Prove that the nilradical of a ring A is the intersection of all the prime ideals of

(b) If I_i, I_j are coprime ideals whenever $i \neq j$. Prove that $\prod I_i = \cap I_i$.

19. (a) Prove that the ring $S^{-1}A$ and the homomorphism $f : A \to S^{-1}A$ have the following properties :

    (i) $s \in S \Rightarrow f(s)$ is a unit in $S^{-1}A$;

    (ii) $f(a) = 0 \Rightarrow$ as = 0 for some $s \in S$;

    (iii) Every element of $S^{-1}A$ is of the form $f(a)\, f(s)^{-1}$, for some $a \in A$ and some $s \in S$.

    Conversely prove that these three conditions determine the ring $S^{-1}A$ upto isomorphism.

    (b) If M_m is a flat A_m-module for each maximal ideal m. Prove that M is a flat A-module.

20. (a) Let B be a ring and A is a subring of B. Prove that the following are equivalent :

    (i) $x \in B$ is integral over A.

    (ii) A[x] is a finitely generated A-module.

    (iii) A[x] is contained in a subring C of B such that C is a finitely generated A-module.

    (iv) There exists a faithful A[x]-module M which is finitely generated as an A-module.

    (b) If $A \subseteq B \subseteq C$ are rings and if B is integral over A and C is integral over B, then prove that C is integral over A.

21. (a) If A is Neotherian. Prove that $A[x_1, x_2, \ldots, x_n]$ Neotherian.

    (b) A ring A is Artin if and only if A is Neotherian and dim A = 0.

(2 × 5 = 10 weightage)