Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2020

(CUCSS)

Mathematics

MT 4E 01—COMMUTATIVE ALGEBRA

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Type)

Answer **all** *questions.*
Each question has weightage 1.

1. Define a principal ideal domain. Give an example.
2. If x is a nilpotent element of a ring A, show that $1+x$ is a unit of A.
3. Define radical of an ideal. Explain it for an ideal in $\mathbb{Z}$.
4. If $0 \rightarrow M \xrightarrow{f} M'$ is an exact sequence, verify whether f is injective.
5. Find $3 \otimes x$ in the tensor product $\mathbb{Z} \otimes (\mathbb{Z}/3\mathbb{Z})$ for any $x \in \mathbb{Z}/3\mathbb{Z}$.
6. If S is a multiplicatively closed subset of a ring A and $0 \in S$ show that $S^{-1}A$ is the zero ring.
7. Write an example of a primary ideal which is not a prime ideal.
8. Define a decomposable ideal. Explain it with an example.
9. If $x \in B$ is integral over A, prove that $A[x]$ is a finitely generated A-module.
10. Define descending chain condition. Explain it with an example.
11. Write an example of a ring which is Artinian but not Noetherian.
12. Verify whether a principal ideal domain is Noetherian.
13. Define dimension of a ring. Find the dimension of $\mathbb{Q}$.
14. Let K be a field and R be a finite dimensional vector space over K. Verify whether R is Noetherian.

(14 × 1 = 14 weightage)

Turn over

Part B (Paragraph Type)

Answer any **seven** *questions from the following ten questions.*
Each question has weightage 2.

15. Let p be a prime number and $k \geq 1$. Find the idempotents in $\mathbb{Z}/(p^k)$

16. Find the nilpotents of $\mathbb{Z}/(12)$.

17. If S is a multiplicatively closed set in a ring A, define $S^{-1}A$. Write the conditions under w becomes a field.

18. If m and n are coprime, verify whether $\mathbb{Z}_m \otimes \mathbb{Z}_n = 0$.

19. Let A be a ring, a be an ideal and M be an A-module. Prove that $(A/a) \otimes M$ is isomor M/aM.

20. If the sequence $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of A-modules and homomor and N is any A-module, then show that the sequence $0 \to Hom(M'',N) \xrightarrow{F}$ $(M,N) \xrightarrow{G} Hom(M',N)$ is exact where F, G are induced by f, g respectively.

21. Let $f: A \to B$ be a homomorphism of rings. Let S be a multiplicatively closed set in the rin $T = f(S)$. Prove that $S^{-1}B$ and $T^{-1}B$ are isomorphic as S^{-1} A-modules.

22. If S is a multiplicatively closed subset of a ring A and q is a p-primary ideal such that $S \cap p \neq$ prove that $S^{-1}q = S^{-1}A$.

23. If A is Noetherian, then show that the polynomial ring A[x] is Noetherian.

24. Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is an exact sequence of A-modules. Prove that M is Art and only if M' and M'' are Artinian.

(7 × 2 = 14 wei

Part C (Essay Type)

Answer any **two** *questions from the following four questions.*
Each question has weightage 4.

25. Let M be a finitely generated A-module and a be an ideal of A contained in the Jacobson ra of A. If aM = M then prove that M = 0.

26. Let S be a multiplicatively closed subset of the ring A. If M, M′, M″ are A-modules such that $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M, prove that $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact at $S^{-1}M$.

27. Let a be a decomposable ideal and let $a = \bigcap_{i=1}^{n} q_i$ be a minimal primary decomposition of a. Let $p_i = r(q_i), 1 \leq i \leq n$. Prove that $p_i = r(a : x)$ for some $x \in A, 1 \leq i \leq n$. Further show that the prime ideals of a are precisely the minimal elements in the set of all prime ideals containing a.

28. (a) Let $A \subseteq B$ be integral domains and B is integral over A. Prove that B is a field if and only if A is a field.

(b) Let $A \subseteq B$ be rings, B is integral over A and p be a prime ideal of A. Prove that there exists a prime ideal q of B such that $q \cap A = p$.

(2 × 4 = 8 weightage)