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THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2021

[November 2020 for SDE/Private Students]

(CBCSS)

Mathematics

MTH 3C 13—FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions (Not applicable to SDE/Private Students)

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section / sub part / sub division may be ignored.*
4. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries weightage 1.

1. Consider the subspace E_1 of a linear space E. Prove that the dimension of E/E_1 is n if and only if there exists linearly independent vectors $x_1, x_2 \ldots x_n$ linearly independent vectors relative to E_1 such that every vector of E can be uniquely expressed as a sum of their linear combination and a unique vector $y \in E_1$.
2. Is C [0, 1] a normed space ? Justify your answer.
3. State Holder's inequality and derive Cauchy Schwartz inequality from the same.
4. Show that inner product $\langle x, y \rangle$ is a continuous function with respect to both variables .
5. Prove that any two separable infinite dimensional Hilbert spaces H_1, H_2 are isometrically equivalent.
6. Let $f \in E^{II} / \{0\}$. Show that *codim ker* $f = 1$.

Turn over

7. State Arzela's theorem.

8. Show that there exists a linear functional on a normed space X that distinguishes distinct el of X.

(8 × 1 = 8 weig

Part B

Answer **six** *questions choosing* **two** *from each unit.*
Each question carries weightage 2.

Unit 1

9. Show that norm is a continuous function.

10. Is a quotient space a normed space ? Justify your answer.

11. Show that the kernel for a seminorm p is a subspace of a linear space on which it is de show that $p(x + y)$ is independent of, where y is an element of the subspace.

Unit 2

12. State Bessel's inequality and use it to show that a complete orthonormal system in a H H is a basis in H.

13. Prove that f is a bounded functional on a normed space X if and only if f is continuous

14. If E is a closed subspace of a Hilbert space H and codim E = 1 then prove that dimensional subspace.

Unit 3

15. Show that l_1 can be identified as the dual space of c_0.

16. Prove that the dual space of any normed space is complete.

17. Let X, Y be any *two* Banach spaces .Prove that for a linear operator $A : X \rightarrow Y$ implies is compact.

(6 × 2 = 1

Part C

Answer **two** *questions.*
Each question carries weightage 5.

18. Let E be a normed space. Show that there exists a complete normed space $\hat{E}$ and linear operator $T : E \to \hat{E}$ such that :

    (i) $\|T(x)\| = \|x\|$.

    (ii) *Im* (T) is a dense set in $\hat{E}$.

19. State and prove a necessary condition for a Hilbert space to have an orthonormal basis.

20. (a) Consider $f \in E^* / \{0\}$. Prove that :

    (i) *codim kerf* = 1.

    (ii) $f, g \in E^{\#} / \{0\}$ and $\ker f = \ker g$ *then there exists* $\lambda \neq 0$ such that $\lambda f = g$.

    (iii) If L is a closed subspace of E and *codim* L = 1 then there exists $f \in E^{\#}$ such that $\ker f = 1$.

    (b) Illustrate with an example the concept of non-separable Hilbert space.

21. (a) Discuss the compactness of the integral operator in L_2.

    (b) State and prove necessary and sufficient condition for a set to be relatively compact in a normed space.

(2 × 5 = 10 weightage)