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Name..

Reg. No.....................................

THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (REGULAR) EXAMINATION NOVEMBER 2020

(CBCSS)

Mathematics

MTH 3C 11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admissions)

Time : Three Hours

Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each Section/Part.*
2. *The minimum number of questions to be attended from the Section/Part shall remain same.*
3. *There will be an overall ceiling for each Section/Part that is equivalent to maximum weightage of the Section/Part.*

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Suppose X is a vector space, and dim X = n. Show that the set E of n vectors in X spans X if and only if E is independent.

2. Show that if I is the identity operator on $\mathbb{R}^n$, then $\det[I] = \det(e_1, e_2, \ldots, e_n) = 1$.

3. Define a tangent vector. Calculate the tangent vector of the curve $\gamma(t) = (\cos^2 t, \sin^2 t)$.

4. Show that, if $f(x, y)$ is a smooth function, its graph $\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$ is a smooth surface with atlas consisting of the single regular surface patch

   $\sigma(u, v) = (u, v, f(u, v))$.

5. Show that, if $\sigma(u, v)$ is a surface patch, the set of linear combinations of σ_u and σ_v is unchanged when σ is reparametrized.

6. Calculate the first fundamental forms of the surface $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.

7. Show that $\|\sigma_u \times \sigma_v\| = \left(\|\sigma_u\|\|\sigma_v\| - (\sigma_u \cdot \sigma_v)^2\right)^{1/2}$.

8. Show that the second fundamental form of a plane is zero.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **six** *questions.*
Each question has weightage 2.

9. Show that a linear operator on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

10. Show that if $A,B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then $\|A+B\| \le \|A\| + \|B\|, \|cA\| \le |c|\|A\|$ Also show that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with the distance between A and B defined as $\|A-B\|$.

11. Prove that det([B][A]) = det[B]det[A], for nxn matrices [A] and [B].

12. Compute the curvature of the circular helix $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta); \theta \in \mathbb{R}$.

13. Define a surface. Show that any open disc in the xy-plane is a surface.

14. Define orientable surface. Show that Mobius band is not orientable.

15. Show that $\sigma(u,v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$ is a regular surface patch for S^2.

16. Show that Gaussian curvature of a ruled surface is negative or zero.

17. Calculate the principal curvatures of the torus

    $\sigma(\theta,\varphi) = ((a+b\cos\theta)\cos\varphi, (a+b\cos\theta)\sin\varphi, b\sin\theta)$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. State and prove the Inverse function theorem.

19. Show that f is a local diffeomorphism if and only if, for all $p \in S$, the linear map $D_p f : T_p S \to T_{f(p)} \tilde{S}$ is invertible, where S and $\tilde{S}$ are surfaces and $f : S \to \tilde{S}$ a smooth map.

20. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ be a unit-speed curve, let $S_0 \in (\alpha, \beta)$ and let φ_0 be such that $\dot{\gamma}(s_0) = (\cos\varphi_0, \sin\varphi_0)$. Then there is a unique smooth function $\varphi : (\alpha,\beta) \to \mathbb{R}$ such that $\varphi(s_0) = \varphi_0$ and $\dot{\gamma}(s) = (\cos\varphi(s), \sin\varphi(s))$ holds for all $s \in (\alpha, \beta)$.

21. Prove that a smooth map $f : S_1 \to S_2$ is a local isometry if and only if the symmetric bilinear forms $\langle , \rangle_p$ and $f^* \langle , \rangle_p$ on $T_p S_1$ are equal for all $p \in S_1$.

(2 × 5 = 10 weightage)