D 90737 (Pages : 3) Name..

Reg. No..

THIRD SEMESTER M.A./M.Sc./M.Com. DEGREE (SUPPLEMENTARY) EXAMINATION, NOVEMBER 2020

(CUCSS)

Mathematics

MT 3C 14—FUNCTIONAL ANALYSIS

(2016 Admissions)

Time : Three Hours

Maximum : 36 Weightage

Part A

Answer **all** *the questions.*

Each question carries weightage 1.

1. Let $V = C[-1,1]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$ for $f, g \in V$. Let $f(x) = x$, $g(x) = \frac{3x^2 - 1}{2}$. Check whether f and g are orthogonal ?

2. Show that the metric space l^∞ is not separable.

3. State the finite intersection property.

4. Show that for $p \geq 1$, the sequence space $l^p \subset l^\infty$.

5. Define the quotient norm on the quotient space X/Y, where Y is a closed subspace of a normed space X.

6. Show by an example that a linear map on a linear space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X.

7. Is norm a bounded linear functional on a normed space X ? Justify your answer.

8. State the Hahn-Banach extension theorem.

9. State the Taylor-Foguel Theorem.

10. Is C_{00}, the space of all scalar sequences having only finitely many non-zero terms as a subspac l^{∞}, a closed set ? Justify your answer.

11. State the Bounded inverse theorem.

12. Is the parallelogram equality satisfied in l^1 ? Justify your answer.

13. Give an example of a discontinuous linear map.

14. State the Parseval formula in a Hilbert space.

(14 × 1 = 14 weight

Part B

Answer any **seven** *questions.*
Each question carries weightage 2.

15. Give an example to show that the open mapping theorem may not hold true if the normed sp X and Y are not Banach spaces.

16. Show that the set of all polynomials in one variable is dense in $C[a, b]$ with sup metric.

17. Define a strictly convex normed space. Give an example of a space which is not strictly conve

18. Show that a linear map F from a normed space X to a normed space Y is a homeomorphism if th are $\alpha, \beta > 0$ such that

$$\beta \| x \| \leq \| F(x) \| \leq \alpha \| x \|$$

for all $x \in X$.

19. State and prove the Bessel's inequality.

20. Show that a Banach space cannot have a denumerable Hamel basis.

21. Let X and Y be normed spaces. If Z is a closed subspace of X, then show that the quotient ma from X to X/Z is continuous and open.

22. State and prove the Riesz-Fischer theorem.

23. If all functionals vanish on a given vector, then show that the vector must be zero.

24. Let X be a Banach space, Y a normed linear space and $T_n \in BL(X, Y)$ such that the sequence $(T_n(x))$ is Cauchy in Y for every $x \in X$. Show that the sequence $(\| T_n \|)$ is bounded.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage 4.

25. Consider the norms $\| x \|_1 = | x_1 | + | x_2 | + ... + | x_n |$ and $\| x \|_\infty = \max \{| x_1 |, | x_2 |, ..., | x_n |\}$ for for $x = (x_1, x_2, ..., x_n) \in R^n$.

   a) Prove that $\| x \| = \frac{1}{3} \| x \|_1 + \frac{2}{3} \| x \|_\infty$ defines a norm on R^n.

   b) Sketch the open unit ball in R^2 with respect to the norm in part (a).

   c) In $C[0, 1]$ with supremum norm, compute $d(f, g)$ for $f(x) = 1$ and $g(x) = x$.

26. State and prove the closed graph theorem.

27. Let H be a non-zero Hilbert space over K. Then show that the following conditions are equivalent.

   a) H has a countable orthonormal basis.

   b) H is linearly isometric to K^n for some n or to l^2.

   c) H is separable.

28. Let X and Y are normed spaces and $X \neq 0$. Then show that :

   (a) $BL(X, Y)$, the set of all bounded linear maps from X to Y is a Banach space in the operator norm if and only if Y is a Banach space.

   (b) The dual X' of every normed space X is a Banach space.

(2 × 4 = 8 weightage)