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D 31177 **(Pages : 3)** **Name.....................................**

Reg. No.....................................

THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2022

(CBCSS)

(November 2021 Session for SDE/Private Students)

Mathematics

MTH3C11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Define dimension of a vector space. Show that dim $\mathbb{R}^n = n$.

2. Show that $\det[A]_1 = -\det[A]$, if $[A]_1$ is an $n \times n$ matrices obtained from [A] by interchanging two columns.

3. Define a parametrized curve. Find the parametrization for the level curve $y^2 - x^2 = 1$.

4. Find the signed curvature of the catenary $\gamma(t) = (t, \cosh t)$.

5. Find the equation of the tangent plane of the surface patch :

   $\sigma(r, \theta) = \left(r \cosh\theta, r \sinh\theta, r^2\right)$ at the point (1, 0, 1).

6. Show that $x^2 + y^2 + z^4 = 1$ is a smooth surfaces.

7. Calculate the first fundamental forms of the surface :

   $\sigma(u, v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$.

8. Compute the second fundamental form of the elliptic paraboloid $\sigma(u, v) = \left(u, v, u^2 + v^2\right)$.

(8 × 1 = 8 weightage)

Turn over

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Part B

Answer **six** *questions choosing two from each unit.*
Each question has weightage 2.

Unit I

9. Show that if a vector space X is spanned by a set of r vectors, then $\dim X \leq r$.

10. Show that, if f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, then $f \in \mathscr{C}^1(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m, 1 \leq j \leq n$.

11. If [A] and [B] are n by n matrices, then show that $\det([B][A]) = \det[B]\det[A]$.

Unit II

12. Show that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

13. Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then, show that the image of γ is contained in a plane if and only if the torsion τ is zero at every point of the curve.

14. Calculate the transition map Φ between the two surface patches for the Mobius band. Show that it is defined on the union of two disjoint rectangles in $\mathbb{R}^2$, and that the determinant of the Jacobian matrix of Φ is equal to +1 on one of the rectangles and to −1 on the other.

Unit III

15. Prove that the area of a surface patch is unchanged by reparametrization.

16. Show that the normal curvature of any curve on a sphere of radius r is $\pm 1/r$.

17. Calculate the principal curvatures of the helicoid $\sigma(u, v) = (v \cos u, v \sin u, \lambda u)$.

$(6 \times 2 = 12$ weightage)

Part C

Answer **two** *questions.*
Each question has weightage 5.

18. State and prove the Implicit function theorem.

19. (a) Show that the transition maps of a smooth surface are smooth.

(b) Let U and $\tilde{U}$ be open subsets of $\mathbb{R}^2$ and let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : \tilde{U} \to U$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : U \to \tilde{U}$. Then show that $\tilde{\sigma} = \sigma \circ \Phi : \tilde{U} \to \mathbb{R}^3$ is a regular surface patch.

20. Let S be a subset of $\mathbb{R}^3$ with the following property : for each point $p \in S$, there is an open subset W of $\mathbb{R}^3$ containing p and a smooth function $f : W \to \mathbb{R}$ such that :

(i) $S \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$;

(ii) The gradient ∇f of f does not vanish at p.

Then, show that S is a smooth surface.

21. Let S be a connected surface of which every point is an umbilic. Then, prove that S is an open subset of a plane or a sphere.

(2 × 5 = 10 weightage)