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Name.......................................

Reg. No.......................................

SECOND SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION JUNE 2020

(CBCSS)

Mathematics

MT 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admissions)

Time : Three Hours

Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Find the indicial equation and its roots for the equation :

   $x^3y'' + (\cos 2x - 1)y' + 2xy = 0.$

2. Show that $F'(a, b, c, x) = \frac{ab}{c} F(a+1, b+1, c+1, x)$.

3. Obtain the recursion formula for Legendre polynomials :

   $(n+1)\, p_{n+1}(x) = (2n+1)\, x\, p_n(x) - n\, p_{n-1}(x).$

4. Prove that $\int x^{-p} J_{p+1}(x)\, dx = -x^{-p} J_p(x) + c.$

5. Describe the relation between the phase portraits of the systems :

   $\frac{dx}{dt} = F(x, y), \frac{dy}{dt} = G(x, y)$ and $\frac{dx}{dt} = -F(x, y), \frac{dy}{dt} = -G(x, y).$

6. Show that the function $E(x, y) = ax^2 + bxy + cy^2$ is negative definite iff $a < 0$ and $b^2 - 4ac < 0$.

7. Find the normal form of Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0.$

Turn over

8. Find the stationary function of $\int_0^4 \left[xy' - (y')^2\right] dx$ which is determined by the boundary conditions $y(0) = 0, y(4) = 3.$

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each of the following 3 units.*
Each question carries 2 weightage.

Unit I

9. Determine the recursion formula for the equation $y'' + \left(p + \frac{1}{2} - \frac{1}{4}x^2\right)y = 0$, where p is a constant.

10. Show that Gauss's hypergeometric equation $x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$ has precisely three regular singular points.

11. Find the first three terms of the Legendre series of :

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ x & \text{if } \ 0 \le x \le 1 \end{cases}.$$

Unit II

12. Prove that the positive zeros of $J_p(x)$ and $J_{p+1}(x)$ occur alternately, in the sense that between each pair of consecutive positive zeros of either there is exactly one zero of the other.

13. Determine the nature and stability properties of the critical point $(0, 0)$ for the system :

$$\frac{dx}{dt} = -4x - y, \quad \frac{dy}{dt} = x - 2y.$$

14. Investigate the stability properties of the critical point $(0, 0)$ for the Van der Pol equation

$$\frac{d^2x}{dt^2} + \mu\left(x^2 - 1\right)\frac{dx}{dt} + x = 0, \mu < 0.$$

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Unit III

15. Describe Picard's method of successive approximations for solving the initial value problem $y' = f(x, y), y(x_0) = y_0$.

16. State and prove Sturm Comparison theorem.

17. Find the curve of fixed length L that joins the points $(0, 0)$ and $(1, 0)$, lies above the x-axis, and encloses the maximum area between itself and the x-axis.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 *weightage.*

18. (a) Find two independent Frobenius series solutions of the equation $x^2y'' - x^2y' + (x^2 - 2)y = 0$.

(b) Find the general solution of the equation $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0$ near its singular point $x = 3$.

19. (a) State and prove the orthogonality property for Bessel functions.

(b) If $f(x) = x^p$ for the interval $0 \leq x < 1$, show that its Bessel series in the functions $J_p(\lambda_n x)$, where the λ_n's are the positive zeros of $J_p(x)$, is $x^p = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{p+1}(\lambda_n)} \cdot J_p(\lambda_n x)$.

20. (a) Find the general solution of the system : $\frac{dx}{dt} = -4x - y, \ \frac{dy}{dt} = x - 2y$.

(b) Show that $(0, 0)$ is an asymptotically stable critical point for the system :

$$\frac{dx}{dt} = -2x + xy^3, \ \frac{dy}{dt} = -x^2y^2 - y^3.$$

Turn over

21. (a) Let $f(x, y)$ be a continuous function that satisfies a Lipschitz condition $|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$ on a strip defined by $a \leq x \leq b$ and $-\infty < y < \infty$. If (x_0, y_0) is any point of the strip, then the initial value problem $y' = f(x, y), y(x_0) = y_0$ has one and only one solution $y = y(x)$ on the interval $a \leq x \leq b$.

(b) Show that $f(x, y) = xy^2$ satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$ and $c \leq y \leq d$.

(2 × 5 = 10 weightage)