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Reg. No...................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2024

(CBCSS)

Mathematics

MTH 2C 07—REAL ANALYSIS—II

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Verify whether $\{\emptyset, [0,1], \mathbb{R}\}$ is a σ-algebra over $\mathbb{R}$.

2. Let $E = \cup_n E_n$ where $E_n = \left(\frac{1}{n}, 3 - \frac{1}{n}\right)$. Find $m(E)$.

3. Verify whether $f(x) = 2x$ is a Lebesgue measurable function on $\mathbb{R}$

4. Consider

$$f(x) = \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } 1 \le x \le 2 \end{cases}.$$

Verify whether $f(x)$ is a simple function.

5. Let $f(x)$ in $[0,1]$ be defined by

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ otherwise} \end{cases}.$$

Find $\int_E f$ where E is the set of all rationals in $[0,1]$.

Turn over

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6. Consider the family of functions

$$F = \{\chi_A : A \text{ is a measurable subset of } [0,1]\}.$$

Verify whether F is uniformly integrable.

7. Let $(q_n)_{n=1}^{\infty}$ be an enumeration of the rationals in $(0,1)$ and let

$$f(x) = \sum_{n : q_n \le x} \frac{1}{2^n}.$$

for $0 < x < 1$. Show that f is an increasing function on $(0,1)$.

8. Verify whether $f(x) = \sqrt{x}$ is absolutely continuous on $[0,1]$.

(8 × 1 = 8

Part B

Answer any **two** *questions from each module.*
Each question has weightage 2.

MODULE I

9. Let m^* be the Lebesgue measure on $\mathbb{R}$. Show that for subsets A, B $A \subseteq B$ then $m^*(A) \le m^*(B)$.

10. Show that if A and B are disjoint measurable subsets of $\mathbb{R}$ then

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

11. Show that Lebesgue measure is translation invariant.

MODULE II

12. Let ϕ and ψ be simple functions on a measurable set E of finite measure. Show that

$$\int_E (\phi + \psi) = \int_E \phi + \int_E \psi.$$

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13. Let f be a bounded measurable function on a set E of finite measure. Let $E = A \cup B$ where A and B are disjoint measurable subsets of E. Show that

$$\int_E f = \int_A f + \int_B f.$$

14. Let a sequence (f_n) converge to f in measure on E. Show that there exists a subsequence (f_{n_k}) converging pointwise to f a.e. on E.

Module III

15. Let f be a function of bounded variation on $[0,1]$. Show that f is the differenece of two increasing functions.

16. Show that if f is a Lipschitz function on $[0,1]$. then it is absolutely continuous on $[0,1]$.

17. Show that every convergent sequence in a normed linear space is a Cauchy sequence.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Define Lebesgue outer measure of a subset A of the reals.

(b) Show that $m^*([a, b]) = b - a$ where $[a, b]$ is a closed interval

(c) Show that $m^*((a, b)) = b - a$ where (a, b) is an open interval

19. (a) Define Lebesgue measurable function.

(b) Show that every continuous function defined on a measurable subset of $\mathbb{R}$ is a measurable function.

(c) Show that if f is measurable and if $g = f$ a.e. then g is measurable.

20. (a) Define tight family of functions.

(b) Let (f_n) be a sequence of functions defined on a measurable set E that is unifor[m]

and tight over E. Suppose that $f_n \to f$ pointwise a.e. on E. Show that :

(i) f is integrable over E.

(ii) $\int_E f = \lim_{n \to \infty} \int_E f_n$.

21. (a) Define the space $L^1(E)$ for a measurable set E.

(b) Show that $\| f \|_1 = \int_E |f|$ is a norm on $L^1(E)$.

(c) Show that $L^1(E)$ is a Banach space.

(2 x 5 =