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ECOND SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, APRIL 2024

Mathematics

MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admission onwards)

ne : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Define (i) regular and (ii) singular points of a differential equation.

2. If $p(x)$ is a polynomial of degree $n \geq 1$ such that

$$\int_{-1}^{1} x^k p(x)dx = 0 \text{ for } k = 0, 1, \ldots, n-1,$$

show that $p(x) = cP_n(x)$ for some constant c.

3. Determine whether the following function is positive definite, negative definite, or neither :

$$2x^2 - 3xy + 3y^2.$$

4. Find general solution of the system :

$$\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = 4x - 2y. \end{cases}$$

5. Describe the phase portrait of the system :

$$\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -y. \end{cases}$$

6. Show that $f(x, y) = xy$ satisfies a Lipschitz condtion on any rectangle $a \leq x \leq b$ and $c \leq y \leq d$.

7. Describe Picard's iteration method.

8. Give the solution, if exists to the initial value problem, $|y'| + |y| = 0$, with the initial condition $y(0) = 1$.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **two** *questions from each of the 3 units.*
Each question carries a weightage of 2.

Unit I

9. Let x_0 be an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

and let a_0 and a_1 be arbitrary constants. Then prove that there exists a uniq[illegible] $y(x)$ that is analytic at x_0 and is a solution of the differential equation in neighbourhood of this point, and satisfies the initial conditions $y(x_0) = a_0$ and [illegible] Also prove that if the power series expansions of $P(x)$ and $Q(x)$ are valid on a[illegible] $|x - x_0| < R, R > 0$, then the power series expansion of this solution is also valid o[illegible] interval.

10. Locate and classify singular points on the x-axis of the differential equation :

$$x^2(x-1)y'' - 2(x-1)y' + 3xy = 0.$$

11. Find the general solution of the differential equation

$$\left(x^2 - 1\right)y'' + (5x + 4)y' + 4y = 0$$

near the singular point $x = -1$.

Unit II

12. If the two solutions

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \text{ and } \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$$

of the homogeneous system

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$$

are linearly independent on $[a, b]$, and if

$$\begin{cases} x = x_p(t) \\ y = y_p(t) \end{cases}$$

is any particular solution of

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases}$$

on this interval, then prove that

$$\begin{cases} x = c_1x_1(t) + c_2x_2(t) + x_p(t) \\ y = c_1y_1(t) + c_2y_2(t) + y_p(t) \end{cases}$$

is the general solution of

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t) \end{cases}$$

on $[a, b]$.

13. Prove that the function $E(x, y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$, and is negative definite if and only if $a < 0$ and $b^2 - 4ac < 0$.

14. Determine the nature and stability properties of the critical point (0, 0) for the following linear autonomous system :

$$\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = 5x + 2y. \end{cases}$$

Unit III

15. Let $y(x)$ be a nontrivial solution of the differential equation $y'' + q(x)y = 0$ on a closed interval $[a, b]$. Then prove that $y(x)$ has at most a finite number of zeros in this interval.

16. Find approximate solutions by Picard's iteration method to the initial value problem $y' = 1 + y^2$ with the initial condition $y(0) = 0$. Also compare with exact solution.

17. Find the curve of fixed length L that joins the points (0, 0) and (1, 0), lies above the x-axis, and encloses the maximum area between itself and the x-axis.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) Find a power series solution of the form $\sum a_n x^n$ of the differential equation $xy' = y$.

(b) Assume that $x = 0$ is a regular singular point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad (1)$$

Turn over

and that the power series expansions

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

of $xP(x)$ and $x^2Q(x)$ are valid on an interval $|x| < R$ with $R > 0$. Let the indicial

$$m(m-1) + mp_0 + q_0 = 0$$

have real roots m_1 and m_2 with $m_2 \leq m_1$. Then prove that the equation (1) ha
one solution

$$y_1 = x^{m1} \sum_{n=0}^{\infty} a_n x^n \ (a_0 \neq 0)$$

on the interval $0 < x < R$, where the a_n are determined in terms of a_0 by the formula

$$a_n \left[(m+n)(m+n-1) + (m+n)p_0 + q_0\right] + \sum_{k=0}^{n-1} a_k \left[(m+k)p_{n-k} + q_{n-k}\right] = 0$$

with m replaced by m_1 and the series $\sum a_n x^n$ converges for $|x| < R$. Also pro
$m_1 - m_2$ is not zero or a positive integer, then equation (1) has a second ind
solution

$$y_2 = x^{m_2} \sum_{n=0}^{\infty} a_n x^n \ (a_0 \neq 0)$$

on the same interval, where in this case the a_n are determined in terms of a_0 b
(2) with m replaced by m_2, and again the series $\sum a_n x^n$ converges for $|x| <$

19. (a) If there exists a Liapunov function $E(x, y)$ for the system

$$\begin{cases} \dfrac{dx}{dt} = F(x, y) \\ \dfrac{dy}{dt} = G(x, y) \end{cases}$$

then prove that the critical point (0, 0) is stable. Also, if this function has the a
property that the function

$$\frac{\partial E}{\partial x} F + \frac{\partial E}{\partial y} G$$

is negative definite, then prove that the critical point (0, 0) is asymptotically

(b) Obtain $J_p(x)$, the Bessel function of first kind of order p.

20. (a) Find the general solution of the system

$$\frac{dx}{dt} = 4x - 2y, \frac{dy}{dt} = 5x + 2y.$$

(b) For the following nonlinear system :

(i) Find the critical points ;

(ii) Find the differential equation of the paths ;

(iii) Solve this equation to find the paths

$$\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = 2x^2y^2 \end{cases}$$

21. (a) Find the stationary function of

$$\int_0^4 \left[xy' - (y')^2 \right] dx$$

which is determined by the boundary conditions $y(0) = 0$ and $y(4) = 3$.

(b) State and prove Picard's theorem.

(2 × 5 = 10 weightage)