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Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2022

April 2021 Session for SDE/Private Students

(CBCSS)

Mathematics

MTH 2C 06—ALGEBRA—II

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

Covid Instructions are not applicable for SDE/Private Students

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section / sub part / sub division may be ignored.*
4. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Let R be a finite commutative ring with unity. Show that every prime ideal in R is a maximal ideal.
2. Show that $\sqrt{1+\sqrt[3]{2}}$ is algebraic over $\mathbb{Q}$.
3. Find a basis for $Q\left(\sqrt{2},\sqrt{3}\right)$ over $\mathbb{Q}$.
4. Find the number of primitive 8th roots of unity in GF (9).
5. What is the order of $G\left(Q\left(\sqrt[3]{2},i\sqrt{3}\right)/Q\left(\sqrt[3]{2}\right)\right)$?
6. Let E be a finite extension of the field F. Show that if E is separable over F then each α in E is separable over F.

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7. Show that the regular 7– gon is not constructible.

8. Find the 6th cyclotomic polynomial over the rational Q.

(8 × 1 = 8 w

Part B

Answer any **two** *questions from each of the following 3 units.*
Each question carries 2 weightage.

Unit I

9. Prove that $Q\left(2^{1/2}, 2^{1/3}\right) = Q\left(2^{1/6}\right)$.

10. Prove that a field F is algebraically closed if and only if every non-constant polynom factors in F [x] into linear factors.

11. Show that trisecting the angle is impossible.

Unit II

12. Let E be a field of p^n elements contained in an algebraic closure $\overline{\mathbb{Z}}_p$ of $\mathbb{Z}_p$. Show that th of E are precisely the zeros in $\mathbb{Z}_p$ of the polynomial $x^{p^n} - x$ in $\mathbb{Z}_p[x]$.

13. Let $\overline{F}$ be algebraic closure of F and $E \leq \overline{F}$ be a splitting field over F. Prove that every is of E into $\overline{F}$ leaving elements of F fixed is an automorphism of E.

14. Prove that If E is a finite extension of F, then {E : F} divides [E : F].

Unit III

15. State the Main Theorem of Galois Theory.

16. Show that Galois group of the n^{th} cyclotomic extension of Q has $\phi(n)$ elements and is i to the group consisting of the positive integers less than n and relatively prime t multiplication modulo n.

17. Let $y_1, y_2, \ldots, y_5$ be independent transcendental real numbers over Q. Show that the p $f(x) = \prod_{i=1}^{5}(x - y_i)$ is not solvable by radicals over $F = Q(s_1, s_2, \ldots, s_5)$, where s_i is the i^{th} e symmetric function in $y_1, y_2, \ldots, y_5$.

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Part C

Answer any **two** *questions.*
Each question carries 5 weightage.

8. (a) Let R be a commutative ring with unity. Prove that M is a maximal ideal of R if and only if R/M is a field.

   (b) Find a non-trivial proper ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not prime.

9. (a) Let E be an algebraic extension of a field F. Show that there exist a finite number of elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ in E such that $E = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ if and only if E is a finite extension of F.

   (b) Let E be an extension of a field F. Show that the set G (E/F) of all automorphisms of E leaving F fixed is a group under function composition and the fixed field of G (E/F) contains F.

10. (a) Let F be a field, and let α and β be algebraic over F with $\deg(\alpha, F) = n$. Prove that the map $\psi_{\alpha\beta} : F(\alpha) \to \beta$ defined by $\psi_{\alpha\beta}\left(c_0 + c_1\alpha + \ldots + c_{n-1}\alpha^{n-1}\right) = c_0 + c_1\beta + \ldots + c_{n-1}\beta^{n-1}$ for $c_i \in F$ is an isomorphism of $F(\alpha)$ onto $F(\beta)$ if and only if α and β are conjugate over F.

    (b) Find the splitting field of the polynomial $x^4 + x^2 + 1$ in $Q[x]$.

11. Let F be a field of characteristic 0, and let $a \in F$. Prove that if K is the splitting field of $x^n - a$ over F, then G (K/F) is a solvable group.

(2 × 5 = 10 weightage)