

1749

(Pages : 3)

Name.....

Reg. No.....

**SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY)
EXAMINATION, APRIL 2021**

(CBCSS)

Mathematics

MT 2C 07—REAL ANALYSIS—II

(2019 Admissions)

Time : Three Hours

Maximum : 30 Weightage

General Instructions

1. In cases where choices are provided, students can attend **all** questions in each section.
2. The minimum number of questions to be attended from the Section / Part shall remain the same.
3. There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.

Part A

Answer all questions.

Each question carries a weightage of 1.

1. Let A be the set of irrational numbers in the interval $[0, 1]$, Prove that $m^*(A) = 1$.
2. Prove that a monotone function that is defined on an interval is a measurable function.
3. If $\{f_k\}_{k=1}^n$ is a finite family of measurable functions with common domain E , then prove that the functions $\max \{f_1, \dots, f_n\}$ and $\min \{f_1, \dots, f_n\}$ are also measurable.
4. Let f and g be bounded measurable functions on a set of finite measure E . If $f \leq g$ on E , then prove that $\int_E f \leq \int_E g$.
5. Let E be a set of finite measure and let $\delta > 0$ be given. Prove that E is the disjoint union of a finite collection of sets, each of which has measure less than δ .
6. If $\{f_n\} \rightarrow f$ in measure on E , then prove that there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f .

Turn over

7. Find the upper and lower derivatives of f at $x = 0$ for the function $f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.
8. Give an example of a Cauchy sequence of real numbers that is not rapidly Cauchy.

Part B

Answer any six questions by choosing two questions from each unit.
Each question carries a weightage of 2.

UNIT I

9. Show that every interval is a Borel set.
10. Show that the Cantor set is an uncountable set of measure zero.
11. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise to a function f . Prove that f is measurable.

UNIT II

12. Show that the function f defined on $[0, 1]$ by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational, is not Riemann integrable over $[0, 1]$, but it is Lebesgue integrable over $[0, 1]$.
13. State and prove the Monotone Convergence theorem.
14. Let E have finite measure, $\{f_n\} \rightarrow f$ in measure on E and g is a measurable function on E finite a.e. on E . Prove that $\{f_n \cdot g\} \rightarrow f \cdot g$ in measure.

UNIT III

15. Let f and g be real-valued functions on (a, b) . Show that, on (a, b) ,
- $$\underline{D}f + \underline{D}g \leq \underline{D}(f + g) \leq \bar{D}(f + g) \leq \bar{D}f + \bar{D}g.$$
16. State and prove Jensen's Inequality.
17. Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^p(E)$, then prove that their sum $f + g$ also belong to $L^p(E)$. Also prove that

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

(6 × 2 = 12)

Part C

Answer any **two** questions.
Each question carries a weightage of 5.

Define the outer measure $m^*(A)$ of a set $A \subset \mathbb{R}$ and give an example. Prove that the outer measure of an interval is its length.

- (a) State and prove Egoroff's theorem.
- (b) Let f be a bounded function on a set of finite measure E . Prove that f is Lebesgue integrable over E if and only if it is measurable.
- (a) Let E have measure zero. Show that if f is a bounded function on E , then f is measurable and $\int_E f = 0$.
- (b) Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E . If $\{f_n\} \rightarrow f$ pointwise a.e. on E , then prove that f is integrable over E and $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$.
- (a) If the function f is monotone on the open interval (a, b) , then prove that it is differentiable almost everywhere on (a, b) .
- (b) Prove that a function f defined on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

(2 × 5 = 10 weightage)