51 (Pages : 3) Name.................................

Reg. No.................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2021

(CBCSS)

Mathematics

MT 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admissions)

Three Hours Maximum : 30 Weightage

General Instructions

In cases where choices are provided, students can attend **all** *questions in each section.*

The minimum number of questions to be attended from the Section / Part shall remain the same.

There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Show that $x = 1$ is a regular singular point of the equation $x^2(x^2-1)^2 y'' - x(1-x)y' + 2y = 0$.

2. Show that $\cos x = \lim_{a \to \infty} F\left(a, a, \frac{1}{2}, \frac{-x^2}{4a^2}\right)$.

3. Use Rodrigue's formula to obtain $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

4. Show that $J_{-m}(x) = (-1)^m J_m(n)$ for any non-negative integer m.

5. Describe the phase portrait of the system :

$$\frac{dx}{dt} = x, \frac{dy}{dt} = 0.$$

6. Show that (0, 0) is an asymptotically stable critical point for the system :

$$\frac{dx}{dt} = -3x^3 - y, \frac{dy}{dt} = x^5 - 2y^3.$$

Turn over

7. Verify whether $f(x,y) = y^{\frac{1}{2}}$ satisfies a Lipschitz condition on the rectangle $|x| \le 1$, ...

8. Find the external for the integral $I = \int_{0}^{1} \left[y^2 - (y')^2 \right] dx$.

Part B

Answer any **two** *questions from each of the following 3 units.*
Each question carries 2 weightage.

Unit I

9. Express $\sin^{-1}(x)$ in the form of a power series by solving the equation $y' = \left(1 - x^2\right)^{-\frac{1}{2}}$ ... two ways.

10. Determine the nature of the point $x = \infty$ for Legendre's equation $\left(1 - x^2\right)y'' - 2xy' + ...$ where p is a constant.

11. Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$. Determine the polynomial ... degree n that minimizes the integral $I = \int_{-1}^{1} \left[f(x) - p(x) \right]^2 dx$.

Unit II

12. Show that between any two positive zeros $J_0(x)$ there is a zero of $J_1(x)$ and that between two positive zeros of $J_1(x)$ there is a zero of $J_0(x)$.

13. Determine the nature and stability properties of the critical point $(0, 0)$ for the system $\frac{dx}{dt} = -3x + 4y, \frac{dy}{dt} = -2x + 3y$.

14. Show that $(0, 0)$ is a simple critical point for the system $\frac{dx}{dt} = x + y - 2xy, \frac{dy}{dt} = -2x + y + ...$ determine its nature and stability properties.

Unit III

15. Consider the initial value problem $y' = 2x(1 + y), y(0) = 0$, starting with $y_0(x) = 0$, apply ... method to calculate $y_1(x), y_2(x), y_3(x)$.

16. Let $u(x)$ be any non-trivial solution of $u'' + q(x)u = 0$, where $q(x) > 0$ for all $x > 0$. Show that if $\int_1^{\infty} q(x)\,dx = \infty$, then $u(x)$ has infinitely many zeros on the positive x-axis.

7. A curve in the first quadrant joins $(0, 0)$ and $(1, 0)$ and has a given area beneath it. Show that the shortest such curve is an arc of a circle.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 weightage.

8. (a) Solve Legendre's equation $(1 - x^2)y'' - 2xy' + p(p+1)y = 0$, where p is a constant.

(b) Show that equation $4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0$ has only one Frobenius series solution and find it.

9. (a) Derive Rodrigue's formula for Legendre polynomials ; $P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}(x^2 - 1)^n$.

(b) Obtain $J_p(x)$, the Bessel function of first kind of order p.

(a) Find the general solution of the system :

$$\frac{dx}{dt} = 4x - 2y, \frac{dy}{dt} = 5x + 2y.$$

(b) Find the critical points and the differential equation of the paths of the system :

$$\frac{dx}{dt} = y(x^2 + 1); \frac{dy}{dt} = 2xy^2.$$

(a) Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is any interior point of R, then show that there exists a number $h > 0$ with the property that the initial value problem $y' = f(x, y), y(x_0) = y_0$ has one solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

(b) Show that if $y(x)$ is a non-trivial solution of $y'' + q(x)y = 0$, then $y(x)$ has an infinite number of positive zeros if $q(x) > \frac{k}{x^2}$ for some $k > \frac{1}{4}$.

(2 × 5 = 10 weightage)