D 32718 (Pages : 3) Name............................

Reg. No............................

FIRST SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, NOVEMBER 2022

Mathematics

MTH 1C 03—REAL ANALYSIS—I

(2019 Admission onwards)

Time : Three Hours Maximum Weightage : 30

Part A (Short Answer Questions)

Answer **all** *questions.*
Each question carries a weightage 1.

1. Prove that countable union of countable set is countable.
2. Let f be a continuous real function on a metric space X. Let Z(f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that Z(f) is closed.
3. Let f be defined for all real x, and suppose that $|f(x) - f(y)| \le (x - y)^2$ for all real x and y. Prove that f is constant.
4. Prove that lower Riemann-Stieltjes integral is always less than or equal to the upper integral.
5. If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$, then prove that $fg \in \mathcal{R}(\alpha)$.
6. Suppose f is a bounded real function on $[a, b]$ and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Justify.
7. Show that the limit of the integral need not be equal to the integral of the limit.
8. Define an equicontinuous family of functions. How equicontinuity and uniform continuity are related ?

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions, choosing* **two** *questions from each unit.*
Each question carries a weightage 2.

Unit I

9. Let X be an infinite set. For $p \in X$ and $q \in X$, define $d(p,q) = \begin{cases} 1 & (\text{if } p \ne q) \\ 0 & (\text{if } p = q). \end{cases}$ Prove that this is a metric. Which subsets of the resulting metric space are open ? Which are closed ? Which are compact ?

Turn over

Lp

34

10. Prove that a mapping f of a metric space X into a metric space Y is continuous on X only if $f^{-1}(V)$ is open in X for every open set V in Y.

11. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. that $f(X)$ is compact.

Unit II

12. Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$.

13. Prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exist a partition P such $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

14. State and prove the linearity and monotonicity properties of Riemann-Stieltjes inte

Unit III

15. Define uniform convergence of a series of functions. State and prove the Weiestrass for uniform convergence of a series of functions.

16. Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit of E and su that $\lim_{t \to x} f_n(t) = A_n (n = 1, 2, 3, \ldots)$. Then prove that $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to x} A$

17. If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

(6 × 2 = 12 weigh

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. (a) Construct a compact set of real numbers whose limit points form a countable set

(b) Let f and g be complex continuous functions on a metric space X. Prove that $f + g$ and f/g are also continuous on X.

19. (a) State and prove Taylor's theorem.

(b) Prove that continuity is a sufficient condition for Riemann-Stieltjes integrability.

20. (a) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n($ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, prove $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n \to x} f_n'(x)$ $(a < x < b)$.

(b) Show that for every interval $[-a, a]$, there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that $\lim_{n \to \infty} P_n(x) = |x|$ uniformly on $[-a, a]$.

21. (a) Prove that $\mathcal{C}(X)$, the set of all complex valued, continuous, bounded functions defined on a metric space X is a complete metric space.

(b) If K is a compact metric space, if $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, prove that $\{f_n\}$ is equicontinuous on K.

(2 × 5 = 10 weightage)