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Reg. No..............................

FIRST SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, NOVEMBER 2022

Mathematics

MTH1C02—LINEAR ALGEBRA

(2019 Admission onwards)

Time : Three Hours

Maximum Weightage : 30

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Let $\mathbb{C}$ be the field of complex numbers. Find the vectors in C^3 spanned by the vectors $(1, 0, -1)$, $(0, 1, 1)$ and $(1, 1, 1)$.
2. Let 0 be the zero vector of a vector space V. Prove that $v0 = 0$ for all $v \in V$.
3. Let S be a non-empty subset S of a vector space V. Prove that S is a subspace of V if $\alpha u + \beta v \in S$ for every $u, v \in S$ and for every scalar α, β.
4. Check whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (x - y, 0)$ is a linear transformation or not.
5. In the Euclidean space $\mathbb{R}^3$ with standard innerproduct, prove that the vectors $(1, 0, 1)$, $(-1, 0, 1)$ and $(0, 1, 0)$ is an orthogonal set of vectors.
6. Prove that the similar matrices have same characteristic polynomials.
7. Let $W_1, \ldots, W_2$ be subspaces of a vector space V. Prove that W_1 and W_2 are linearly independent if and only if $W_1 \cap W_2 = \{0\}$.
8. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be element in $\mathbb{R}^2$. Prove that $(\alpha | \beta) = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2$ is an inner product on $\mathbb{R}^2$.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *by choosing* **two** *questions from each unit.*
Each question carries 2 *weightage.*

Unit I

9. Let $\mathbb{F}$ be a field. prove that $\mathbb{F}^n$ is a vector space.
10. Let S be a subset of a vector space V. Prove that a subspace spanned by S is the set of all linear combinations of elements in S.

Turn over

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11. If A is an $n \times n$ matrix over $\mathbb{F}$ and B, C are $n \times p$ matrices over $\mathbb{F}$. Prove that for any k, $A(kB + C) = kAB + AC$.

Unit II

12. Prove or disprove: For any subset S of a finite dimensional vector space V, $(S^{\circ})^{\circ}$ is subspace spanned by S.
13. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined as projection to the first component. Let $\mathcal{B}$ be the standard basis of $\mathbb{C}^2$ and let $\mathcal{B}' = \{(1, i), (-i, 2)\}$. Find the matrix relative to the pair, $\mathcal{B}$, $\mathcal{B}'$.
14. Let W_1 and W_2 be subspaces of a finite dimensional vector space. Prove that $W_1 = W_2$ if and only $W_1^{\circ} = W_2^{\circ}$.

Unit III

15. Prove that an orthogonal set of non-zero vectors is linearly independent.
16. Let V be a finite dimensional vector space. Let $W_1, \ldots, W_k$ be subspaces of V and let $\mathcal{B}_i$ be an ordered basis for each W_i for $i = 1, \ldots, k$. If $W_1, \ldots W_k$ is linearly independent prove that the sequence $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ is an ordered basis for $W_1 + \ldots + W_k$.
17. If W_1 be any subspace of a finite dimensional vector space V then prove the existence of a subspace W_2 of V such that $V = W_1 \oplus W_2$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 *weightage.*

18. Let V be a vector space which is spanned by a finite set of vectors $v_1, \ldots, v_m$. Prove that any independent set of vectors in V is finite and contains no more than m elements.
19. Let V be a vector space over a field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field and let $\beta_1, \beta_2, \ldots, \beta_n$ be any n vectors in W. Prove that these exist a unique linear transformation $T : V \to W$ such that $T_{\alpha_i} = \beta_i$, $i = 1, 2, \ldots, n$.
20. Let θ be a real number. Prove that the following two matrices are similar over $\mathbb{C}$.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, B = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

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21. Let T be a linear operator on a finite dimensional vector space V, Prove that the minimal polynomial for T divides its characteristic polynomial. Let T be a linear operator on a vector space $V = W_1 \oplus \ldots \oplus W_k$. Then prove that there exist projections E_i on V such that

(a) $T = c_1 E_1 + \ldots + c_k E_k$.

(b) $I = E_1 + \ldots + E_k$.

(c) $E_i E_j = 0, i \neq j$ and

(d) The range of E_i is the characteristic space for T that is associated with c_i.

($2 \times 5 = 10$ weightage)