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**D 90232** (Pages : 4) Name..................................

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# FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2020

(CUCBCSS—UG)

Mathematics

## MAT 5B 07—BASIC MATHEMATICAL ANALYSIS

Time : Three Hours Maximum : 120 Marks

### Section A

*Answer* **all** *questions.*

*Each question carries* 1 *mark.*

1. Find $f \circ g$ for the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x, g(x) = 3x^2 - 1$.

2. Is there exists a bijection between $\mathbb{N}$ and a proper subset of itself ? Justify.

3. State the principle of Strong Induction.

4. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.

5. Define absolute value of a real number.

6. Find $\sup\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$.

7. Prove that $\lim \frac{1}{n} = 0$.

8. Give an example of two divergent sequences X and Y such that their sum X + Y converges.

9. Give an example of an unbounded sequence that has a convergent subsequence.

10. Define an open subset of $\mathbb{R}$.

11. Find Re $z$ and Im $z$ for $z = \frac{2+i}{(1+i)(1-2i)}$.

12. Show that $\text{Re}(iz) = -\text{Im}\, z$.

(12 × 1 = 12 marks)

**Turn over**

## Section B

*Answer at least* **eight** *questions.*
*Each question carries 6 marks.*
*All questions can be attended.*
*Overall Ceiling 48.*

13. Let $f : A \to B$ be a function and let $G, H \subseteq B$. Prove that $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

14. Prove that if $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$; $a, b, c \in \mathbb{R}$.

15. Prove that set $\mathbb{N} \times \mathbb{N}$ is denumerable.

16. Prove that a sequence of real numbers can have at most one limit.

17. Prove that every convergent sequence of real numbers is bounded.

18. Prove that the sequence $(n)$ is divergent.

19. Prove that $\lim \left(\frac{1}{n^2+1}\right) = 0$.

20. Prove that every convergent sequence of real numbers is a Cauchy sequence.

21. Prove or disprove : The arbitrary intersection of open sets in $\mathbb{R}$ is open.

22. Show that if G is an open set and F is a closed set, then $G|F$ is an open set and $F|G$ is a closed set.

23. Show that $|e^{i\theta}| = 1$.

24. Prove that $z$ is real if and only if $\bar{z} = z$.

25. Sketch the given set and determine whether it is a domain : $|2z + 3| > 4$.

26. Define accumulation point of a set. Determine the accumulation points, if any, for the set $z_n = i^n$; $n = 1, 2, 3, \ldots$.

(8 × 6 = 48 marks)

## Section C

*Answer at least* **five** *questions.*

*Each question carries* 9 *marks.*

*All questions can be attended.*

*Overall Ceiling* 45.

27. If $I_n = [a_n, b_n]; n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, prove that there exists a real number $\xi$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.

28. Prove that the set $\mathbb{R}$ of real numbers is not countable.

29. State and prove Monotone Convergence Theorem.

30. State and prove Bolzano-Weierstrass Theorem.

31. Let $(x_n)$ and $(y_n)$ be sequences of real numbers such that $x_n \leq y_n \ \forall n \in \mathbb{N}$. Prove that :

    (a) If $\lim (x_n) = +\infty$, then $\lim (y_n) = +\infty$.

    (b) If $\lim (y_n) = -\infty$, then $\lim (x_n) = -\infty$.

32. Let $F \subseteq \mathbb{R}$. Prove that the following are equivalent :

    (a) F is a closed subset of $\mathbb{R}$.

    (b) If $X = (x_n)$ is any convergent sequence of elements in F, then $\lim X = x$ belongs to F.

33. Prove that a subset of $\mathbb{R}$ is closed if and only if it contains all of its cluster points.

34. Let $z$ be a complex number. Prove that $|1 + z| = 1 + |z|$ if and only if $z$ is real.

35. Find all roots of $8^{1/6}$ in rectangular co-ordinates.

(5 × 9 = 45 marks)

## Section D

*Answer any* **one** *question.*
*The question carries* 15 *marks.*

36. Prove that there exists a positive real number $x$ such that $x^2 = 2$.

37. Let $X = (x_n)$ be a sequence of real numbers and let $x \in \mathbb{R}$. Prove that the following are equivalent :

    (a) X converges to $x$.

    (b) For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$ the terms $x_n$ satisfy $|x_n - x| < \varepsilon$.

    (c) For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$ the terms $x_n$ satisfy $x - \varepsilon < x_n < x + \varepsilon$.

    (d) For every $\varepsilon$-neighborhood $V_\varepsilon(x)$ of $x$, there exists a natural number K such that for all $n \geq K$ the terms $x_n$ belong to $V_\varepsilon(x)$.

38. Prove that a subset of $\mathbb{R}$ is open if and only if it is the union of countably many disjoint open intervals in $\mathbb{R}$.

(1 × 15 = 15 marks)